Combinatorial Optimisation, Exam 20 October 2009

The examination consists of three pages containing five exercises. The maximum number of points to be gained on the various parts are displayed in the following table:

1a	1b	2	3a	3b	4a	4b	5a	5b	5c
5	5	10	8	2	5	5	4	4	2

The result is obtained by dividing the total number of points by 5. This implies that 28 points are needed to pass.

During the examination only the book Comb. Opt. by Pap. & Steigl. *without leaves is allowed to be on your desk and all electronic equipment should be switched off.*

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1. Given a complete graph on 8 vertices, numbered $1, 2, \ldots, 8$. In de following table the distances between the vertices are given. Distances are symmetric, i.e., the distance from vertex i to vertex j is equal to the distance from vertex j to vertex i. Therefore it is sufficient to know the numbers above the diagonal.

	1	2	3	4	5	6	7	8
1	0	7	9	5	11	16	18	20
2		0	4	3	4	10	15	17
3			0	6	6	7	11	15
4				0	7	13	13	15
5					0	6	12	13
6						0	7	8
7							0	6
8								0

(a) Find the shortest path from vertex 1 to vertex 8 using Dijkstra's algorthm. Indicate explicitly the labels assigned to the vertices in in each iteration.

(b) Find the minimum spanning tree in the graph. Indicate which algorithm you are using and give as the answer the length of the minimum spanning tree and the sequence of edges chosen consecutively by the algorithm.

2. Indicate for each of the four statements below if it is *true* or *false*. A correct answer will give $2\frac{1}{2}$ points and an incorrect answer gives $-1\frac{1}{2}$. Argumentation of the answers is not needed. A correct argumentation with a wrong answer is wrong.

Given two problems Π_1 and Π_2 that are both member of the class NP:

(a) If Π_1 is NP-Complete and Π_1 is reducible to Π_2, $\Pi_1 \propto \Pi_2$, then Π_2 is NP-Complete;

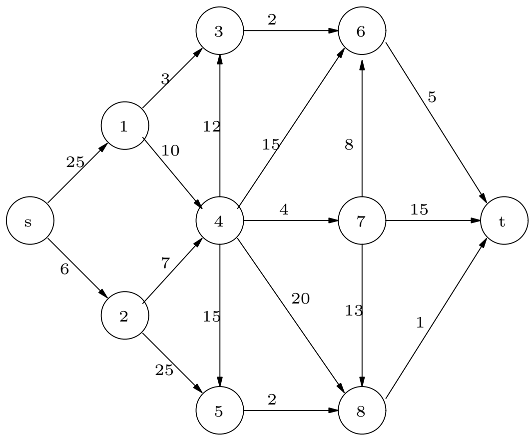
(b) If $\Pi_1 \in P$ and Π_1 is reducible to Π_2, $\Pi_1 \propto \Pi_2$, then $\Pi_2 \in P$;

(c) If $\Pi_1 \in P$ and Π_2 is reducible to Π_1, $\Pi_2 \propto \Pi_1$, then $\Pi_2 \in P$;

(d) If Π_1 is NP-Complete and Π_2 is reducible to Π_1, $\Pi_2 \propto \Pi_1$, then Π_2 is NP-Complete.

3. Consider the network given in the figure below. The number at an arc represent the capacity of that arc.

(a) Compute the maximum flow through this network from node s to node t, using an augmenting path algorithm. Show in each iteration which augmenting path you used.



(b) Determine an s–t cut of minimum capacity in the network. Show how you find this minimum cut.

4. A graph G is *bipartite* if V can be split into two sets V_1 and V_2 such that for each edge $e = \{u, v\} \in E$, we have $|e \cap V_1| = |e \cap V_2| = 1$. Below are two other characterisations of bipartite graphs.

(a) Prove the following theorem.

Theorem 1 *A graph is bipartite if and only if it has no odd cycles.*

(b) Complete the proof of the following theorem. In the theorem the incidence matrix of a graph $G = (V, E)$ is defined as the matrix with each edge $\{i, j\} \in E$ defining a column of the matrix with a 1-entry in the rows corresponding to vertices i and j and a 0-entry everywhere else.

Theorem 2 *The incidence matrix of a graph is Totally Unimodular (TUM)* $\Leftrightarrow$ *the graph is bipartite.*

Proof: $\Leftarrow$. We prove that if a graph is bipartite then the matrix is TUM by induction on the size of the square submatrices. From the description of the matrix above it is clear that all 1×1 submatrices have determinant 0 or 1. Suppose now that all $(k-1) \times (k-1)$ square submatrices have determinants -1, 0 or 1. Consider a $k \times k$ square submatrix B. Finish this part of the proof by considering the following 3 distinct cases:

- *Case 1.* If B contains a column with only 0-entries;
- *Case 2.* B contains a column with exactly one 1-entry;
- *Case 3.* B has in every column 2 1-entries.

$\Rightarrow$. Hint: prove this direction by contradiction using the charaterisation of bipartite graphs in part (a). (Even if you could not prove part (a), you can use the Theorem in (a).)

5. Consider the vertex cover problem. A vertex cover of a graph $G = (V, E)$ is a subset V' of the vertices such that for each edge $\{i, j\} \in E$, $|V' \cap \{i, j\}| \geq 1$.

VERTEX COVER.
Instance: Graph $G = (V, E)$.
Question: Find a vertex cover of minimum cardinality, i.e., with a minimum number of vertices.

(a) Formulate the VERTEX COVER problem as an Integer Linear Programming problem. Use x_i as binary variable to denote if a vertex is in the vertex cover or not.

(b) Consider the following approximation algorithm. Solve the LP-relaxation of the ILP-formulation in (a). Let x^{LP} be the optimal solution of the LP-relaxation. Round this solution in the following way: if $x_i^{LP} \geq \frac{1}{2}$ then set $x_i = 1$; if $x^{LP} < \frac{1}{2}$ then set $x_i = 0$. Prove that this algorithm produces indeed a vertex cover.

(c) We call this algorithm LP-Rounding (LPR). Let $Z^{LPR}(I)$ be the solution value produced by LPR on instance I, and $Z^{OPT}(I)$ be the optimal solution value of instance I. Prove that

$$\max_I \frac{Z^{LPR}(I)}{Z^{OPT}(I)} \leq 2;$$

i.e., the worst-case performance ratio of LPR is at most 2.