Combinatorial Optimisation, Exam 20 October 2009

The examination consists of three pages containing five exercises. The maximum number of points to be gained on the various parts are displayed in the following table:

1a	1b	2	3a	3b	4a	4b	5a	5b	5c
5	5	10	8	2	5	5	4	4	2

The result is obtained by dividing the total number of points by 5. This implies that 28 points are needed to pass.

During the examination only the book Comb. Opt. by Pap. & Steigl. *without leaves is allowed to be on your desk and all electronic equipment should be switched off.*

---

1. Given a complete graph on 8 vertices, numbered $1, 2, \ldots, 8$. In de following table the distances between the vertices are given. Distances are symmetric, i.e., the distance from vertex i to vertex j is equal to the distance from vertex j to vertex i. Therefore it is sufficient to know the numbers above the diagonal.

	1	2	3	4	5	6	7	8
1	0	7	9	5	11	16	18	20
2		0	4	3	4	10	15	17
3			0	6	6	7	11	15
4				0	7	13	13	15
5					0	6	12	13
6						0	7	8
7							0	6
8								0

(a) Find the shortest path from vertex 1 to vertex 8 using Dijkstra's algorthm. Indicate explicitly the labels assigned to the vertices in in each iteration.

(b) Find the minimum spanning tree in the graph. Indicate which algorithm you are using and give as the answer the length of the minimum spanning tree and the sequence of edges chosen consecutively by the algorithm.

Answer (a). The correct answer labels in each iteration the vertices as follows. The underlined labels correspond to the choice of vertex in the iteration for which the shortest path from vertex 1 has been found.

	2	3	4	5	6	7	8
iteration 1	$(7,1)$	$(9,1)$	$\underline{(5,1)}$	$(11,1)$	$(16,1)$	$(18,1)$	$(20,1)$
iteration 2	$\underline{(7,1)}$	$(9,1)$		$(11,1)$	$(16,1)$	$(18,1)$	$(20,1)$
iteration 3		$\underline{(9,1)}$		$(11,1)$	$(16,1)$	$(18,1)$	$(20,1)$
iteration 4				$\underline{(11,1)}$	$(16,1)$	$(18,1)$	$(20,1)$
iteration 5					$\underline{(16,1)}$	$(18,1)$	$(20,1)$
iteration 6						$\underline{(18,1)}$	$(20,1)$
iteration 7							$\underline{(20,1)}$

The optimal solution is the path going directly from 1 to 8 and has length 20.

Answer (b). I choose for Kruskal's algorithm in this answer. Of course, correct implementation of Prim's algorithm is also correct.
Order the edges in non-decreasing length. Choose first $(2,4)$ in the MST, then add $(2,3)$, then add $(2,5)$, then add $(1,4)$, then reject $(3,4)$, then reject $(3,5)$, then add $(5,6)$, then add $(7,8)$, then reject $(1,2)$, then reject $(3,6)$, then reject $(4,5)$, then add $(6,7)$, by which 7 edges have been selected hence a spanning tree has been obtained.

2. Indicate for each of the four statements below if it is *true* or *false*. A correct answer will give $2\frac{1}{2}$ points and an incorrect answer gives $-1\frac{1}{2}$. Argumentation of the answers is not needed. A correct argumentation with a wrong answer is wrong.
Given two problems Π_1 and Π_2 that are both member of the class NP:

(a) If Π_1 is NP-Complete and Π_1 is reducible to Π_2, $\Pi_1 \propto \Pi_2$, then Π_2 is NP-Complete;

(b) If $\Pi_1 \in P$ and Π_1 is reducible to Π_2, $\Pi_1 \propto \Pi_2$, then $\Pi_2 \in P$;

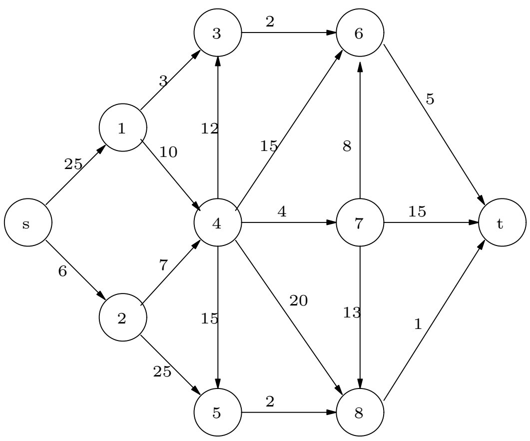
(c) If $\Pi_1 \in P$ and Π_2 is reducible to Π_1, $\Pi_2 \propto \Pi_1$, then $\Pi_2 \in P$;

(d) If Π_1 is NP-Complete and Π_2 is reducible to Π_1, $\Pi_2 \propto \Pi_1$, then Π_2 is NP-Complete.

Answer. (a) TRUE, **(b)** FALSE, **(c)** TRUE, **(d)** FALSE

3. Consider the network given in the figure below. The number at an arc represent the capacity of that arc.

(a) Compute the maximum flow through this network from node s to node t, using an augmenting path algorithm. Show in each iteration which augmenting path you used.



(b) Determine an s–t cut of minimum capacity in the network. Show how you find this minimum cut.

Answer (a). For example: First raise the flow along the path $s-1-4-6-t$ by 5, the capacity of the arc $(5,t)$. In the residual network this arc stops to exist (it's reverse is inserted with capacity 5). Then raise the flow along the path

$s-2-4-7-t$ by 4, the capacity of the arc $(4,7)$. In the residual network this arc stops to exist (it's reverse is inserted with capacity 4). Then raise the flow along the path $s-1-4-8-t$ by 1, the capacity of the arc $(8,t)$. In the residual network this arc stops to exist (it's reverse is inserted with capacity 1). In the residual graph t is no longer reachable from s, hence there are no further augmenting paths and therefore we have found a maximum flow which is $1+4+5=10$.

Answer (b). In the residual graph of the maximum flow, the vertices reachable from s are $1,2,3,4,5,6,8$. This gives the minimum cut: indeed check that the arcs from $\{s,1,2,3,4,5,6,8\}$ to $\{7,t\}$ have capacities $c(5,t)=5$, $c(4,7)=4$ and $c(8,t)=1$, adding up to 10.

4. A graph G is *bipartite* if V can be split into two sets V_1 and V_2 such that for each edge $e=\{u,v\}\in E$, we have $|e\cap V_1|=|e\cap V_2|=1$. Below are two other characterisations of bipartite graphs.

(a) Prove the following theorem.

Theorem 1 *A graph is bipartite if and only if it has no odd cycles.*

(b) Complete the proof of the following theorem. In the theorem the incidence matrix of a graph $G=(V,E)$ is defined as the matrix with each edge $\{i,j\}\in E$ defining a column of the matrix with a 1-entry in the rows corresponding to vertices i and j and a 0-entry everywhere else.

Theorem 2 *The incidence matrix of a graph is Totally Unimodular (TUM) $\Leftrightarrow$ the graph is bipartite.*

Proof: $\Leftarrow$ We prove that if a graph is bipartite then the matrix is TUM by induction on the size of the square submatrices. From the description of the matrix above it is clear that all 1×1 submatrices have determinant 0 or 1. Suppose now that all $(k-1)\times(k-1)$ square submatrices have determinants -1, 0 or 1. Consider a $k\times k$ square submatrix B. Finish this part of the proof by considering the following 3 distinct cases:

- *Case 1.* If B contains a column with only 0-entries;
- *Case 2.* B contains a column with exactly one 1-entry;
- *Case 3.* B has in every column two 1-entries.

$\Rightarrow$ Hint: prove this direction by contradiction using the charaterisation of bipartite graphs in part (a). (Even if you could not prove part (a), you can use the Theorem in (a).)

Answer (a). *G is bipartite $\Rightarrow$ G contains no odd cycle.*
In a bipartite graph $G=(U,V,E)$ every walk starting from a vertex u in U has after each odd number of steps a vertex of V and after each even number of steps a vertex from U. Thus, if the walk returns to u it can do so only after an even number of steps. The same applies to any vertex $v\in V$.

G contains no odd cycle $\Rightarrow$ G is bipartite.
Without loss of generality we assume that the graph is connected. Start in any vertex v of the graph, assign v to V and apply a Breadth-First-Search, yielding the spanning tree T consisting of the shortest (minimum number of edges) paths from v to all the other vertices reachable from v. All vertices add odd distance

from v are assigned to U and all vertices at even distance from v are assigned to V. We claim that this is a proper bipartition of the vertices. If not then there is an edge between two vertices u_1 and u_2 from U (or en edge between two vertices v_1 and v_2 of V). This edge together with the the paths in T from v to u_1 and from v to u_2 would form a odd cycle, which we assumed not to exist. The same holds for the edge (v_1, v_2) together with the paths from v to v_1 and from v to v_2. *qed*

Answer (b). *Proof:* $\Leftarrow$ We prove that if a graph $G = (U, V, E)$ is bipartite then the matrix is TUM by induction on the size of the square submatrices. From the description of the matrix above it is clear that all 1×1 submatrices have determinant 0 or 1. Suppose now that all $(k-1) \times (k-1)$ square submatrices have determinants -1, 0 or 1. Consider a $k \times k$ square submatrix B. Finish this part of the proof by considering the following 3 distinct cases:

- *Case 1.* If B contains a column with only 0-entries,
  In this case, elementary linear algebra tells us that the determinant of B is 0;
- *Case 2.* B contains a column with exactly one 1-entry,
  Let this entry be b_{ij}, the entry in the i-th row of B and the j-th column of B. Elementary linear algebra tells us that $\det(B) = (-1)^{i+j-1}\det(M_{ij})$, where M_{ij} is the $(k-1) \times (k-1)$ matrix obtained by deleting from B the i-th row and the j-th column. By the induction hypothesis $\det(M_{ij}) \in \{-1, 0, +1\}$, hence $\det(B) \in \{-1, 0, +1\}$;
- *Case 3.* B has in every column two 1-entries.
  This implies that each column has one 1-entry in a row corresponding to a vertex from V and one 1-entry in a row corresponding to a vertex from U. This implies that adding all rows of B corresponding to vertices of V yields a vector with all k elements equal to k. The same vector is obtained by adding all rows of B corresponding to vertices of U. Elementary linear algebra tells us that the rows of B are linearly dependent and therefore $\det(B) = 0$.

$\Rightarrow$ Hint: prove this direction by contradiction using the charaterisation of bipartite graphs in part (a). (Even if you could not prove part (a), you can use the Theorem in (a).)
Suppose the incidence matrix is TUM, but the graph G is not bipartite. Then, using the Theorem in (a), it contains an odd cycle C. Let k be the length of C. Take the incidence matrix of C. This is a $k \times k$ square submatrix, B say, of the incidence matrix of G. We prove that this matrix B has determinant equal to 2.

Exchange its rows and columns such that they correspond to the vertices and edges of C encountered in the order when starting from some vertex (row 1) and following C in one of the two directions. Elementary linear algebra tells us that exchanging rows and columns may alter the sign but not the absolute value of the determinant. The matrix has now a 1 for each entry of the diagonal, each entry just below the diagonal and the entry $(1,k)$, and a 0 everywhere else:

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & & & 0 & 1 \\ 1 & 1 & 0 & \cdots & & & 0 & 0 \\ 0 & 1 & \ddots & & & & 0 & 0 \\ 0 & 0 & \ddots & \ddots & & & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & & \ddots & \ddots & 1 & 0 \\ 0 & 0 & 0 & & & \ddots & 1 & 1 \end{pmatrix}.$$

The matrix M_{11} obtained by deleting the first row and column is a lower-triangular matrix (all 0 elements above the diagonal). Since any upper- or lower-triangular matrix has determinant equal to the product of its diagonal elements, $\det M_{11} = 1$. The matrix M_{1k}, obtained by deleting the first row and the last column is upper-triangular and therefore $\det M_{1k} = 1$.

$$\det(B) = 1\det M_{11} + (-1)^{1+k-1}\det M_{1,k}.$$

Since k is odd, $(-1)^{1+k-1} = 1$ and hence $\det(B) = 2$. *qed*

5. Consider the vertex cover problem. A vertex cover of a graph $G = (V, E)$ is a subset V' of the vertices such that for each edge $\{i, j\} \in E$, $|V' \cap \{i, j\}| \geq 1$.

Vertex Cover.
Instance: Graph $G = (V, E)$.
Question: Find a vertex cover of minimum cardinality, i.e., with a minimum number of vertices.

(a) Formulate the Vertex Cover problem as an Integer Linear Programming problem. Use x_i as binary variable to denote if a vertex is in the vertex cover or not.

(b) Consider the following approximation algorithm. Solve the LP-relaxation of the ILP-formulation in (a). Let x^{LP} be the optimal solution of the LP-relaxation. Round this solution in the following way: if $x_i^{LP} \geq \frac{1}{2}$ then set $x_i = 1$; if $x^{LP} < \frac{1}{2}$ then set $x_i = 0$. Prove that this algorithm produces indeed a vertex cover.

(c) We call this algorithm LP-Rounding (LPR). Let $Z^{LPR}(I)$ be the solution value produced by LPR on instance I, and $Z^{OPT}(I)$ be the optimal solution value of instance I. Prove that

$$\max_I \frac{Z^{LPR}(I)}{Z^{OPT}(I)} \leq 2;$$

i.e., the worst-case performance ratio of LPR is at most 2.

Answer (a).

$$\begin{array}{lll} \min & \sum_{i=1}^{n} x_i & \\ \text{s.t.} & x_i + x_j \geq 1, & \forall \{i,j\} \in E \\ & x_i \in \{0,1\}, & \forall i \in V. \end{array}$$

Answer (b). Since for each edge $\{i,j\} \in E$, the optimal solution of the LP-relaxation must satisfy $x_i + x_j \geq 1$, it implies that not both x_i and x_j can have value strictly less than 1/2. Thus at least one of them will be rounded to 1, hence the edge $\{i,j\}$ is covered. *qed*

Answer (c). Clearly, by the rounding for each variable the rounded value is no more than 2 times the optimal value in the LP-relaxation. Thus, using Z^{LP} as the optimal value of the LP-relaxation

$$Z^{LPR}(I) \leq 2Z^{LP} \leq 2Z^{OPT}.$$

qed