Combinatorial Optimisation, Exam 21 March 2011

The examination lasts 2 hours and 45 minutes. Grading will be done before April 2, 2011. Students interested in checking their results can make an appointment by e-mail.

The examination consists of four exercises. The maximum number of points to be gained on the various parts are displayed in the following table:

1a	1b	2a	2b	2c	3a	3b	4
5	5	5	5	5	5	10	10

In Questions 1b and 3a the number of points will never be less than 0.

The result is obtained by dividing the total number of points by 5. This implies that 28 points are needed to pass.

During the examination only the book *Combinatorial Optimization* by Papadimtriou and Steiglitz without any additional leaflets is allowed to be on your desk and all electronic equipment should be switched off.

1a. Prove the following theorem (Theorem 1):

Theorem 1. Every spanning tree of a graph has exactly $n-1$ edges, where n is the number of vertices of the graph.

Hint: You may use Theorem 2 below in a proof by induction. But any other correct proof is also fine.

Theorem 2. Every spanning tree of a graph has at least one leaf (vertex with degree 1).

1b. Decide for each of the following statements if it is "true" or "false" without giving any explanation. A correct answer gives 1 point, a mistake gives -1 point.

(a) There exist graphs with all vertices having different degrees.

(b) A graph with all of its vertices having degree at least k contains a path of length k.

(c) If in a network none of the capacities of the arcs have integer values, then the maximum flow from s to t in the network cannot have integer value.

(d) If in a network none of the capacities of the arcs have integer values, then it can happen that the value of a maximum flow is not equal to that of a minimum cut.

(e) Given a weighted undirected graph, no two edges of which have equal weight. Any such a graph has a unique minimum spanning tree.

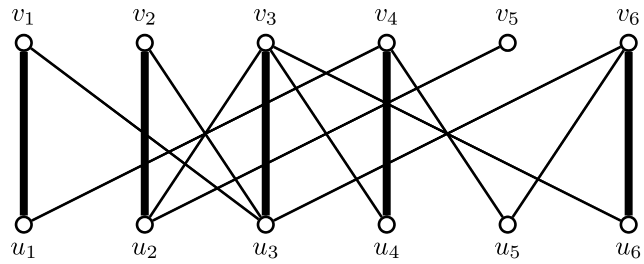

Figure 1: Graph instance of Question 2b. Matching edges are indicated in bold.

2a. Consider any graph $G = (V, E)$. Make a *subdivision* of each of the edges: i.e., for every edge $e = \{u, v\} \in E$ create an extra vertex v_e and replace the edge $\{u, v\}$ by the two edges $\{u, v_e\}$ and $\{v_e, v\}$. (You may think of it as every edge of G getting an extra vertex in the middle.) Prove that the resulting graph is bipartite.

2b. Given is an *unweighted* bipartite graph $G = (V, E)$ displayed in Figure 1. And given is a matching displayed as boldface edges in Figure 1. Starting from this matching determine a maximum matching by using the augmenting path method or conclude that the current matching is optimal.

2c. Theorem 3. In a bipartite graph, the size of a maximum cardinality matching is equal to the size of a minimum cardinality vertex cover.

Prove this theorem.

3a. Decide for each of the following statements if it is "true" or "false" without giving any explanation. A correct answer gives 1 point, a mistake gives −1 point.

(a) If $\Pi_1 \in NP$ and for every $\Pi_2 \in NP$, $\Pi_2 \preceq \Pi_1$, then Π_1 is *NP*-complete.

(b) If $\Pi_1 \in P$ and for every $\Pi_2 \in NP$, $\Pi_2 \preceq \Pi_1$, then $P = NP$.

(c) If $\Pi_1 \preceq \Pi_2$ and $\Pi_2 \preceq \Pi_3$, then $\Pi_1 \preceq \Pi_3$.

(d) If $\Pi_1 \in NP$, Π_2 is *NP*-complete, $\Pi_1 \preceq \Pi_2$, then Π_1 is *NP*-complete.

(e) If Π_1, Π_2 are *NP*-complete, then $\Pi_1 \preceq \Pi_2$ and $\Pi_2 \preceq \Pi_1$.

3b. Prove that the decision version of the Vehicle Routing Problem is *NP*-complete.

Vehicle Routing-decision:

Instance: A set of points X and one central depot c, together with the distance $d(x, y)$ between each pair $x \in X$ and $y \in X \cup \{c\}$. (The distances are symmetric.) The points in X have to be supplied by vehicles from the depot. Each vehicle has a capacity of supplying at most q points. There are sufficiently many vehicles to supply all points. Also given is a constant K.

Goal: Determine whether there exist a set of routes, each starting and ending at the depot c, of total length at most K that supplies all points in X.

Hint: Use the fact that the decision version of the Traveling Salesman Problem *is NP-complete.*

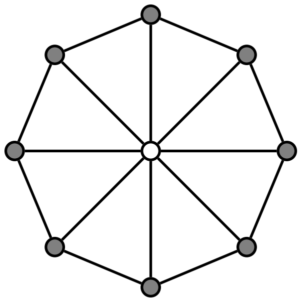

Figure 2: Example instance of the METRIC STEINER TREE PROBLEM in Question 4. There are 8 terminal vertices (indicated in gray) and one Steiner vertex. All edges to the Steiner vertex have cost 1 and all other edges have cost 2 (note that not all edges of the complete graph are shown). Observe that the cost of an optimal Steiner tree is 8, while the cost of a minimum spanning tree on the terminals is 14.

4. Consider the METRIC STEINER TREE PROBLEM:

METRIC STEINER TREE PROBLEM:

Instance:	An undirected complete graph $G = (V, E)$ with non-negative edge costs $(c_e)_{e \in E}$ satisfying the *triangle inequality*, i.e., for every $u, v, w \in V$, $c_{uw} \leq c_{uv} + c_{vw}$, and a set of terminal vertices $R \subseteq V$.
Goal:	Compute a minimum cost tree of G that connects all terminals in R.

The vertices in $V \setminus R$ are called *Steiner* vertices. The METRIC STEINER TREE PROBLEM thus asks for the computation of a minimum cost tree, also called *Steiner tree*, spanning all terminals in R and possibly some Steiner vertices; see Figure 2 for an example.

Develop a 2-approximation algorithm for this problem.

Hint: Show that a minimum spanning tree T on the terminal set R of G satisfies $\mathit{OPT} \geq \frac{1}{2}c(T)$ and use this to derive an approximation algorithm.