Basics of Graph Theory

These notes are a translation of excerpts from Chapter 1 of the Dutch lecture notes *Grafen: Kleuren en Routeren* by Alexander Schrijver (http://homepages.cwi.nl/~lex/files/graphs13.pdf). You may have a look at these notes for pictures, which I am not able to draw in latex. I advise the Dutch students to have a look at the full notes. They may like to read also the Chapters 2 and 3.

Below, the numbering of the Exercises corresponds to the numbering in Schrijver's lecture notes.

A *graph* G is defined by a pair (V,E) where V is a finite set of points and E is a set of pairs of two points. A graph is drawn by depicting the points in V as (thick) dots and the pairs in E as lines between two points. The points defining a graph are called *vertices* and the lines are called *edges*. By $|X|$ we denote the number of elements in the set X.

If for two vertices $u,v \in V$ we have the edge $e = \{u,v\} \in E$ then we say that u and v are *adjacent*. We also say that u and v are *incident* to e and, conversely, that e is incident to u and v. Two edges that share a vertex are also said to be adjacent.

The number of edges incident to vertex $v \in V$ is called the *degree* of v. A graph is called regular if all vertices have the same degree. A graph is called k-regular if all vertices have degree k.

Exercise 1.7. *How many 2-regular graphs exist with* $V = \{1,2,3,4,5\}$*?*

Exercise 1.8. *How many 3-regular graphs exist with* $V = \{1,2,3,4,5\}$*?*

Exercise 1.11. *How many edges has a 5-regular graph on* 16 *vertices?*

Exercise 1.12. *How many edges has a k-regular graph on n vertices?*

Exercise 1.16. *Prove that every graph has an even number of points with odd degree.*

Exercise 1.19. *Prove that a graph* $G = (V,E)$ *with* $|V| = n$ *and* $|E| = m$ *has a vertex with degree* $\leq 2m/n$ *and a vertex with degree* $\geq 2m/n$*.*

Exercise 1.21. *Prove that every graph* $G = (V,E)$ *with* $|V| \geq 2$ *has two vertices of the same degree.*

A graph $G = (V,E)$ *is* complete *if each pair of points is adjacent (defines an edge). A complete graph on n points is denoted by* K_n*.*

Exercise 1.24. *Prove that a graph* $G = (V,E)$ *with* $|V| = n$ *is complete if and only if* G *is* $(n-1)$*-regular.*

A graph G is bipartite *if V can be split into two sets V_1 and V_2 such that for each edge $e = \{u, v\} \in E$, we have $|e \cap V_1| = |e \cap V_2| = 1$. A complete bipartite graph with $V = V_1 \cup V_2$ has edge set $E = \{\{v_1, v_2\} \mid v_1 \in V_1,\ v_2 \in V_2\}$. The complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.*

Exercise 1.26. *For which values of m and n is $K_{m,n}$ regular?*

A walk *in a graph $G = (V, E)$ is a sequence of vertices $(v_0, v_1, \ldots, v_k)$ such that for all $i = 1, \ldots, k$, $\{v_{i-1}, v_i\} \in E$. It is called a walk is* from v_0 to v_k. *We say that this walk has (combinatorial) length k. If all vertices on the walk are distinct we call it a* path.

Exercise 1.39. *How many paths are there from vertex 1 to vertex 3 in K_3?*

Exercise 1.42. *How many paths are there from vertex 1 to vertex n in K_n?*

Exercise 1.43. *Prove that if in a graph there exists a walk from vertex s to vertex t then there exists a path from s to t.*

Exercise 1.44. *Prove that that a graph of which each vertex has degree at least k, has a path of length k.*

A graph is called connected *if there is a path between any two of its vertices.*
Exercise 1.47. *Prove that every connected graph on n vertices contains at least $n-1$ edges.*

Exercise 1.48. *Does there exist a non-connected graph on 6 vertices containing 11 edges.*

Exercise 1.50. *Prove that every non-connected graph on n vertices contains at most $\frac{1}{2}(n-1)(n-2)$ edges.*

A graph $G' = (V', E')$ is called a subgraph *of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. A* component *of $G = (V, E)$ is a maximal connected subgraph $G' = (V', E')$: i.e., if there exists a connected subgraph $\hat{G}$ of G such that G' is a subgraph of $\hat{G}$, then $\hat{G} = G'$. Hence, a graph G is connected if and only if it consists of exactly one component.*

Exercise 1.59. *Prove that if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two distinct components of G then $V_1 \cap V_2 = \emptyset$.*

Exercise 1.61. *Prove that if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two distinct components of G then there exists no edge with one vertex in V_1 and the other in V_2.*

Exercise 1.62. *Prove that there exists a path between u and v if and only if u and v belong to the same component.*

Exercise 1.63. *Prove that a graph $G = (V, E)$ with each vertex having degree at least $\frac{1}{2}(n-1)$ is connected.*

Exercise 1.64. *Prove that a graph $G = (V,E)$ has at least $|V| - |E|$ components.*

Exercise 1.65. *Prove that a graph with exactly two vertices with odd degree must contain a path between these two vertices.*

A walk $(v_0, v_1, \ldots, v_k)$ *is called a* closed walk *or a* cycle *if* $v_0 = v_k$*. A cycle with all vertices distinct is called a* circuit*. A circuit of length* 3 *is called a* triangle.

Exercise 1.67. *Prove that every graph with vertices that each have degree at least* 2 *contains a circuit.*

$G = (V,E)$ *is called a* forest *if G does not contain any circuit. A connected forest is called a* tree.

Exercise 1.74. *Prove that each component of a forest is a tree.*

Exercise 1.75. *Prove that between any pair of vertices in a tree there is exactly one path.*

A vertex of degree 1 *in a tree is called a* leaf *of the tree.*

Exercise 1.76. *Prove that every tree with at least two vertices contains a leaf (cf. Exercise 1.24).*

Exercise 1.77. *Derive from the previous exercise that every tree on n vertices has exactly* $n - 1$ *edges.*

Exercise 1.78. *Prove that every tree with exactly 2 leaves is a path.*

Exercise 1.79. *Prove that a forest on n vertices consisting of k components contains exactly* $n - k$ *edges.*

Exercise 1.80. *Prove that every tree with at least two vertices contains at least two leaves.*

Exercise 1.81. *Let G be a tree with a vertex of degree k. Prove that G contains at least k leaves.*

An Euler cycle *in a graph* $G = (V,E)$ *is a cycle* $C = (v_0, v_1, \ldots, v_k)$ *(remember that* $v_0 = v_k$*), with the property that every edge* $e \in E$ *is traversed exactly once; i.e., for each edge* $e \in E$ *there exists exactly one* $i \in \{1, \ldots, k\}$ *such that* $e = \{v_{i-1}, v_i\}$*. A graph is called an* Euler graph *if it contains an Euler cycle.*

Theorem 1.1 (Euler's Theorem). *A graph* $G = (V,E)$ *is an Euler graph if and only if G is connected and each of its vertices has even degree.*

Exercise. *Prove Euler's Theorem. Try this one before you read it in Schrijver's lecture notes.*

Exercise 1.88. *Let G be a connected graph with exactly two points of odd degree. Use Euler's Theorem to prove that G contains a walk that traverses each edge exactly once.*

A circuit C of G is called a Hamilton circuit *if each vertex of G appears on C. A graph G is called a* Hamilton graph *if it contains a Hamilton circuit.*

Exercise 1.90. *Show that on a $n \times n$ chess-board a horse cannot make a tour visiting each field exactly once if n is odd.*

Exercise 1.91. *Show that for each n there exists a graph on n vertices such that each vertex has degree at least $\frac{1}{2}n - 1$ and such that it is not a Hamilton graph.*

There does not exist a good characterisation of Hamiltonian graphs comparable to Theorem 1.1 for characterising Euler graphs. The next theorem only gives a sufficient condition.

Theorem 1.2 (Dirac's Theorem). *Let G be graph on $n \geq 3$ vertices, each of which has degree at least $\frac{1}{2}n$. Then G is a Hamilton graph.*

PROOF. *Suppose there exists as a counterexample a graph G on n vertices. Then there exists such a graph with a maximum number number of edges. So, G itself is not a Hamilton graph but if we add one edge to G than it becomes a Hamilton graph. Hence, there is a path*

$$P = (v_1, \ldots, v_n)$$

on which every vertex appears exactly once (a so-called Hamilton path. *Because G is not a Hamilton graph, there is no edge between v_1 and v_n. Define*

$$I := \{i \mid v_i \text{ is adjacent to } v_1\};$$
$$J := \{i \mid v_{i-1} \text{ is adjacent to } v_n\}.$$

Because v_1 has degree at least $\frac{1}{2}n$, $|I| \geq \frac{1}{2}n$. Similarly, $|J| \geq \frac{1}{2}n$. Because $I, J \subset \{2, \ldots, n\}$, we have $I \cap J \neq \emptyset$. Choose $i \in I \cap J$. Thus, v_i is adjacent to v_1 and v_{i-1} is adjacent to v_n. Then

$$(v_1, v_i, v_{i+1}, \ldots, v_n, v_{i-1}, v_{i-2}, \ldots, v_2, v_1)$$

is a Hamilton circuit in G, contradicting our assumption. □