Bipartite Matching Exercises

10.1. A set of vertices that covers all edges is called a vertex cover. Given bipartite graph $G = (U, V, E)$ let ν^* be the size of an optimal matching $M^* \subset E$ and τ^* be the size of a vertex cover $V^* \subset \{U, V\}$. Then it is easy to see that $\tau^* \geq \nu^*$, since all ν edges of the optimal matching have to be covered and no two edges in M^* can be covered by the same vertex.

To show that $\tau^* \leq \nu^*$ consider the exposed vertices of matching M^*. Every edge incident to an exposed vertex is adjacent to a matched vertex, because M^* is optimal, hence according to Theorem 10.1 in [P&S] there is no augmenting path w.r.t. M^*. As a vertex cover take all vertices V_{M^*} that are reachable through alternating paths from exposed vertices in U and all vertices U_{M^*} that are reachable through alternating paths from exposed vertices in V. There are no edges in E between vertices of U_{M^*} and V_{M^*}, otherwise we would again have an augmenting path. Thus, together they cover all edges incident to exposed vertices and $|U_{M^*}| + |V_{M^*}|$ edges of the matching M^*. The remaining $\nu^* - (|U_{M^*}|+|V_{M^*}|)$ edges of the matching are covered by choosing their $\nu^* - (|U_{M^*}| + |V_{M^*}|)$ incident U-vertices or their $\nu^* - (|U_{M^*}| + |V_{M^*}|)$ incident V-vertices. In this way we have a vertex cover of size ν^*. Hence $\tau^* \leq \nu^*$. □

An alternative proof is through the LP-formulation:

$$\begin{array}{lll} \max & \sum_{e \in E} x_e & \\ \text{s.t.} & \sum_{e \ni u}^{n} x_e \leq 1, & \forall u \in U; \\ & \sum_{e \ni v}^{n} x_e \leq 1, & \forall v \in V; \\ & x_e \geq 0 & \forall e \in E. \end{array} \qquad (1)$$

We know that the constraint matrix, which is the incidence matrix of a bipartite graph, is totally unimodular, hence there exists an optimal solution of the LP that is integral: 0-1.

The dual is

$$\begin{array}{lll} \min & \sum_{u \in U} y_u + \sum_{u \in U} y_u; & \\ \text{s.t.} & y_u + y_v \geq 1, & \forall e = \{u, v\} \in E; \\ & y_u \geq 0 & \forall u \in U; \\ & y_v \geq 0 & \forall v \in V. \end{array} \qquad (2)$$

and this with integrality constraints added to the dual variables is exactly the vertex cover problem. Since. trivially, the transpose of a totally unimodular matrix is totally unimodular also the LP-relaxation vertex cover problem on a bipartite graph has integral optimal solutions. Strong LP-duality then proves the statement of **10.1**. □

10.2. $\Rightarrow$: This is trivial.

$\Leftarrow$: Proof by contradiction. Let $\Gamma(S) = \{u \in U \mid \exists v \in S : \{v, u\} \in E\}$. Thus $|\Gamma(S) \geq |S| \; \forall S \subset V$. Suppose that given some maximum cardinality matching M there are exposed vertices $U_0 \subset U$ and $V_0 \subset V$. Doing a breadth first search on alternating paths starting in the exposed V_0-wertices fails to end in any exposed vertex of U, otherwise an augmenting path would have been found. Let V_T be all the vertices of $V \setminus V_0$ on the search tree. Let $U_T = \Gamma(V_T \cup V_0)$ be all U-vertices on the search tree. Thus, $U_0 \cap U_T = \emptyset$ and clearly $|U_T| = |V_T|$. Hence, $|\Gamma(V_T \cup V_0)| = |U_T| = |V_T| < |V_T \cup V_0|$, a contradiction. $\square$

10.3. Let $|C^*|$ be the size of an optimal edge cover C^*. That $|C^*| \leq |M| + |V| - 2|M| = |V| - |M|$ for every matching M is easy: a matching M covers $2|M|$ vertices, hence adding for each of the $|V| - 2|M|$ uncovered vertices v some edge $\{v, u\}$ yields an edge cover.

To prove that $|C^*| \geq |V| - |M^*|$, take an optimal edge cover C^*. Such an edge cover will not contain any path of more than two edges, otherwise some edge in the middle may be deleted still keeping an edge cover. Thus the components of C^* form a collection of star-graphs: graphs with one central vertex and all other vertices being leaves (degree 1 vertices). Since any star-graph with k edges covers $k+1$ vertices, there must be $|V| - |C^*|$ star-graph components in C^*. Selecting from each of them one edge yields a matching M. Hence $|M^*| \geq |V| - |C^*|$. $\square$

10.14. For any weight W, if the graph $G(W)$ that consists of only the edges that have weight at most W has a perfect matching M then clearly $\max_{e \in M} w(e) \leq W$. Thus, the algorithm orders the weights of the edges in non-decreasing order and chooses W first equal to the first lowest weight. If the graph $G(W)$ has a perfect matching then this is the weight is the solution value. If not it chooses the next weight in the order. The first weight for which a perfect matching is found is the optimal weight.

11.1. Was done as the example on the blackboard.

11.2. We prove by induction that in each iteration all dual variables are integers or all dual variables have a fractional value $\frac{1}{2}$. Let us start with the dual solution $a_i = 0 \; \forall i \in V$ and $b_j = 0 \; \forall j \in U$, or if you prefer with $a_i = 0 \; \forall i \in V$ and $b_j = \min_{i \in V} c_{ij} \; \forall j \in U$. In both cases the starting solution is all integer.

Suppose our hypothesis is true up to the $k-1$-th iteration. In every iteration, hence also in iteration k, all dual variables are increased or decreased with the same amount

$$\theta = \min_{i \in U^*, j \in V \setminus V^*} \frac{1}{2}(c_{ij} - a_i - b_j). \tag{3}$$

Under our induction hypothesis, at the end of iteration $k-1$, all a_i's and all b_j's are integer or are all fractional $\frac{1}{2}$. In both cases, in iteration k $a_i + b_j$ has integer value, hence θ is a multiple of $\frac{1}{2}$, which added to or distracted from the a_i's and b_j's gives a solution in which the dual solution is again all integer or all fractional $\frac{1}{2}$. $\square$.

11.7. Determine the shortest paths in K_n between all pairs of nodes in S. Define P_{uv} as the the shortest path from u to v, $u, v \in S$ and $d_{uv} = \sum_{\{} i, j\} \in P_{uv} c_{ij}$ as its length. We must assume that $|S|$ is even, otherwise the construction asked for does not even exist. Construct now the complete graph that consists only of nodes in S and all their edges $\{u, v\}$ with weight d_{uv} (so remember the edges actually correspond to paths in the original graph K_n). Then find the minimum weight perfect matching on S and extend all the edges of the matching into the original corresponding paths. Clearly the extension of an edge into the paths gives degree 1 to the end-nodes, which are in S and degree 2 to all intermediate nodes. Hence it is a feasible solution for the problem. Try to prove yourself that it is optimal.

11.8. For the Chinese Postman problem an algorithm is to construct an Eulerian Graph (connected and every vertex even degree) by adding to G a cheapest possible subgraph as in Problem 11.7 on the set $S \subset V$ of all vertices with odd degree in G. This creates in the cheapest possible way an Eulerian graph containing G. Then construct a Eulerian walk on the Eulerian graph, which can be done in polynomial time (find out yourself or in the literature).