State Space Time Series Analysis

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Classical Decomposition

A basic model for representing a time series is the additive model

$$y_t = \mu_t + \gamma_t + \varepsilon_t, \qquad t = 1, \ldots, n,$$

also known as the Classical Decomposition.

$$\begin{aligned} y_t &= \text{observation}, \\ \mu_t &= \text{slowly changing component (trend)}, \\ \gamma_t &= \text{periodic component (seasonal)}, \\ \varepsilon_t &= \text{irregular component (disturbance)}. \end{aligned}$$

In a *Structural Time Series Model (STSM)* or *Unobserved Components Model (UCM)*, the RHS components are modelled explicitly as stochastic processes.

Local Level Model

- Components can be deterministic functions of time (e.g. polynomials), or stochastic processes;
- Deterministic example: $y_t = \mu + \varepsilon_t$ with $\varepsilon_t \sim \mathcal{NID}(0, \sigma^2_\varepsilon)$.
- Stochastic example: the Random Walk plus Noise, or *Local Level* model:

$$
\begin{aligned}
y_t &= \mu_t + \varepsilon_t, \qquad & \varepsilon_t &\sim \mathcal{NID}(0, \sigma^2_\varepsilon) \\
\mu_{t+1} &= \mu_t + \eta_t, \qquad & \eta_t &\sim \mathcal{NID}(0, \sigma^2_\eta),
\end{aligned}
$$

- The disturbances ε_t, η_s are independent for all s, t;
- The model is incomplete without a specification for μ_1 (note the non-stationarity):

$$\mu_1 \sim \mathcal{N}(a, P)$$

Local Level Model

$$
\begin{aligned}
y_t &= \mu_t + \varepsilon_t, & \varepsilon_t &\sim \mathcal{NID}(0, \sigma_\varepsilon^2) \\
\mu_{t+1} &= \mu_t + \eta_t, & \eta_t &\sim \mathcal{NID}(0, \sigma_\eta^2), \\
\mu_1 &\sim \mathcal{N}(a, P)
\end{aligned}
$$

- The level μ_t and the irregular ε_t are unobserved;
- Parameters: $\sigma_\varepsilon^2, \sigma_\eta^2$;
- Trivial special cases:
  - $\sigma_\eta^2 = 0 \implies y_t \sim \mathcal{NID}(\mu_1, \sigma_\varepsilon^2)$ (WN with constant level);
  - $\sigma_\varepsilon^2 = 0 \implies y_{t+1} = y_t + \eta_t$ (pure RW);
- Local Level is a model representation for EWMA forecasting.

Local Linear Trend Model

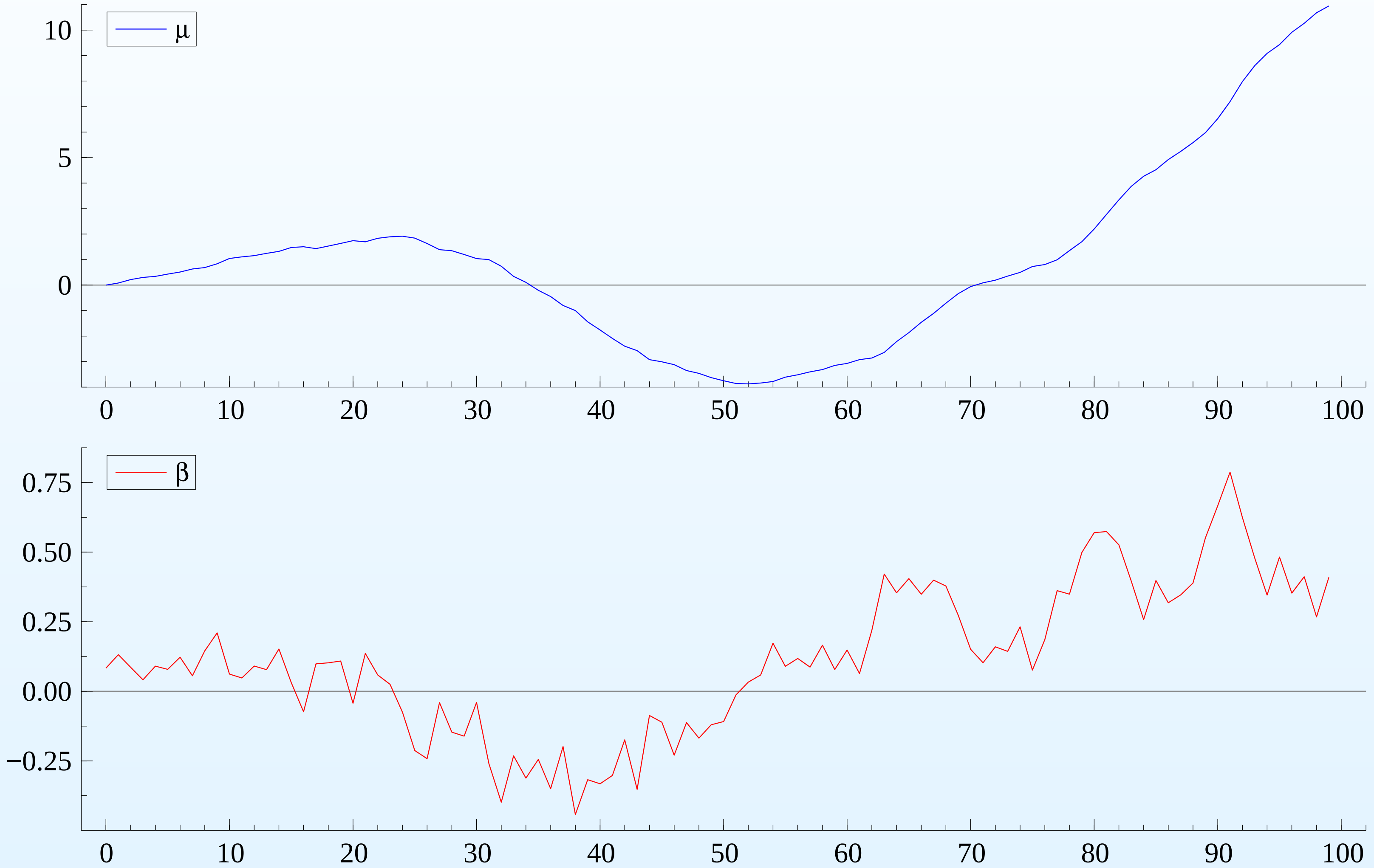
The LLT model extends the LL model with a slope:

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, & \varepsilon_t &\sim \mathcal{NID}(0, \sigma^2_\varepsilon), \\ \mu_{t+1} &= \beta_t + \mu_t + \eta_t, & \eta_t &\sim \mathcal{NID}(0, \sigma^2_\eta), \\ \beta_{t+1} &= \beta_t + \xi_t, & \xi_t &\sim \mathcal{NID}(0, \sigma^2_\xi). \end{aligned}$$

- All disturbances are independent at all lags and leads;
- Initial distributions β_1, μ_1 need to specified;
- If $\sigma^2_\xi = 0$ the trend is a random walk with constant drift β_1; (For $\beta_1 = 0$ the model reduces to a LL model.)
- If additionally $\sigma^2_\eta = 0$ the trend is a straight line with slope β_1 and intercept μ_1;
- If $\sigma^2_\xi > 0$ but $\sigma^2_\eta = 0$, the trend is a smooth curve, or an Integrated Random Walk;

Trend and Slope in LLT Model

Trend and Slope in Integrated Random Walk Model



Local Linear Trend Model

- Reduced form of LLT is ARIMA(0,2,2);
- LLT provides a model for Holt-Winters forecasting;
- Smooth LLT provides a model for spline-fitting;
- Smoother trends: higher order Random Walks

$$\Delta^d \mu_t = \eta_t$$

Seasonal Effects

We have seen specifications for μ_t in the basic model

$$y_t = \mu_t + \gamma_t + \varepsilon_t.$$

Now we will consider the seasonal term γ_t. Let s denote the number of 'seasons' in the data:

- $s = 12$ for monthly data,
- $s = 4$ for quarterly data,
- $s = 7$ for daily data when modelling a weekly pattern.

Dummy Seasonal

The simplest way to model seasonal effects is by using dummy variables. The effect summed over the seasons should equal zero:

$$\gamma_{t+1} = -\sum_{j=1}^{s-1} \gamma_{t+1-j}.$$

To allow the pattern to change over time, we introduce a new disturbance term:

$$\gamma_{t+1} = -\sum_{j=1}^{s-1} \gamma_{t+1-j} + \omega_t, \qquad \omega_t \sim \mathcal{NID}(0, \sigma_\omega^2).$$

The expectation of the sum of the seasonal effects is zero.

Trigonometric Seasonal

Defining γ_{jt} as the effect of season j at time t, an alternative specification for the seasonal pattern is

$$\gamma_t = \sum_{j=1}^{[s/2]} \gamma_{jt},$$

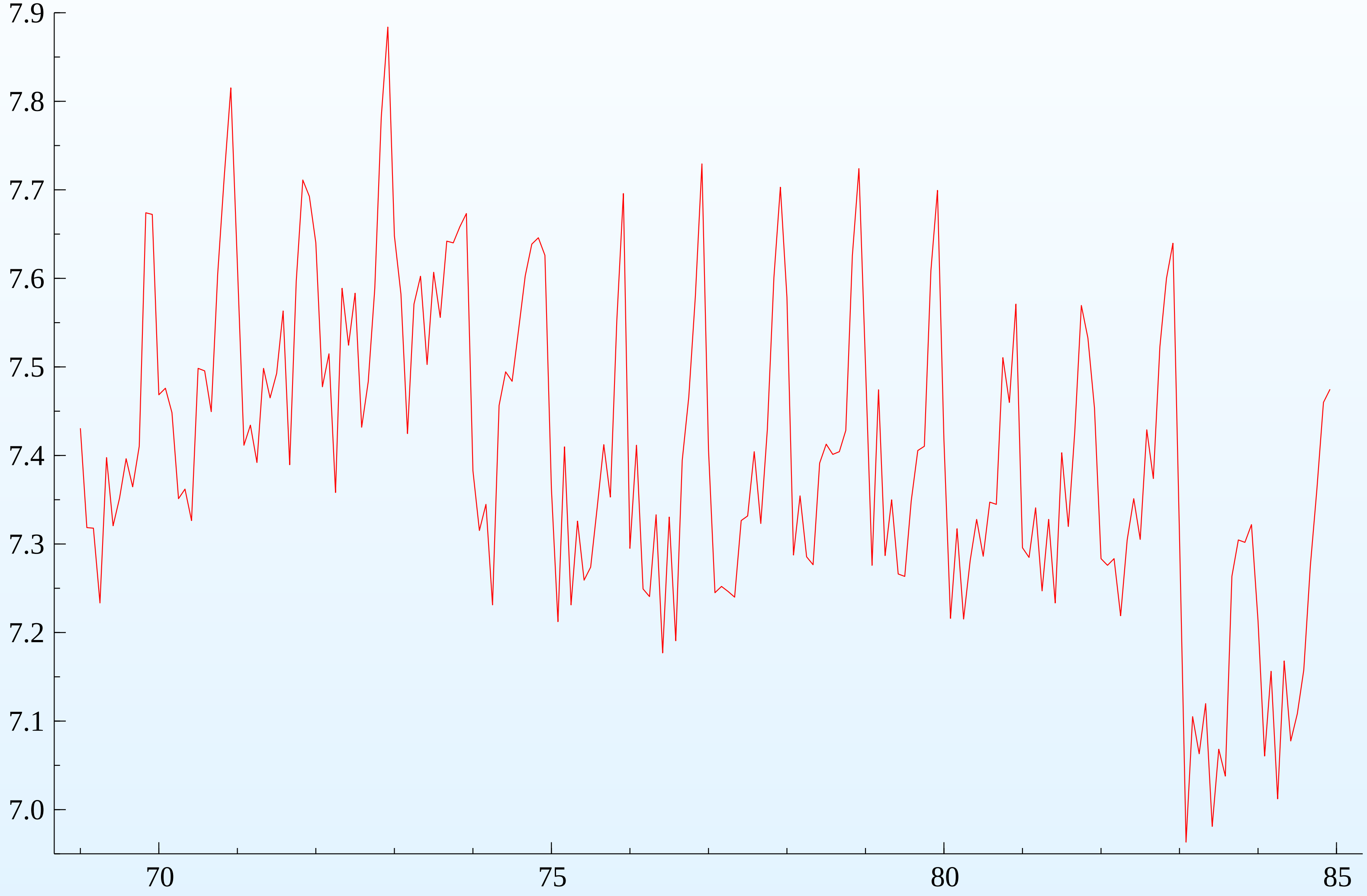
$$\gamma_{j,t+1} = \gamma_{jt} \cos\lambda_j + \gamma_{jt}^* \sin\lambda_j + \omega_{jt},$$

$$\gamma_{j,t+1}^* = -\gamma_{jt} \sin\lambda_j + \gamma_{jt}^* \cos\lambda_j + \omega_{jt}^*,$$

$$\omega_{jt}, \omega_{jt}^* \sim \mathcal{NID}(0, \sigma_\omega^2), \qquad \lambda_j = 2\pi j/s.$$

- Without the disturbance, the trigonometric specification is identical to the deterministic dummy specification.
- The autocorrelation in the trigonometric specification lasts through more lags: changes occur in a smoother way;

Seatbelt Law



State Space Model

Linear Gaussian state space model (LGSSM) is defined in three parts:

$\rightarrow$ State equation:

$$\alpha_{t+1} = T_t \alpha_t + R_t \zeta_t, \qquad \zeta_t \sim \mathcal{NID}(0, Q_t),$$

$\rightarrow$ Observation equation:

$$y_t = Z_t \alpha_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, H_t),$$

$\rightarrow$ Initial state distribution $\alpha_1 \sim \mathcal{N}(a_1, P_1)$.

Notice that

- ζ_t and ε_s independent for all t, s, and independent from α_1;
- observation y_t can be multivariate;
- state vector α_t is unobserved;
- matrices T_t, Z_t, R_t, Q_t, H_t determine structure of model.

State Space Model

- state space model is linear and Gaussian: therefore properties and results of multivariate normal distribution apply;
- state vector α_t evolves as a VAR(1) process;
- system matrices usually contain unknown parameters;
- estimation has therefore two aspects:
  - measuring the unobservable state (prediction, filtering and smoothing);
  - estimation of unknown parameters (maximum likelihood estimation);
- state space methods offer a *unified approach* to a wide range of models and techniques: dynamic regression, ARIMA, UC models, latent variable models, spline-fitting and many ad-hoc filters;
- next, some well-known model specifications in state space form ...

Regression with Time Varying Coefficients

General state space model:

$$\alpha_{t+1} = T_t \alpha_t + R_t \zeta_t, \qquad \zeta_t \sim \mathcal{NID}(0, Q_t),$$
$$y_t = Z_t \alpha_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, H_t).$$

Put regressors in Z_t,

$$T_t = I, \qquad R_t = I,$$

Result is regression model with coefficient α_t following a random walk.

ARMA in State Space Form

Example: AR(2) model $y_{t+1} = \phi_1 y_t + \phi_2 y_{t-1} + \zeta_t$, in state space:

$$\alpha_{t+1} = T_t \alpha_t + R_t \zeta_t, \qquad \zeta_t \sim \mathcal{NID}(0, Q_t),$$
$$y_t = Z_t \alpha_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, H_t).$$

with 2×1 state vector α_t and system matrices:

$$Z_t = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad H_t = 0$$
$$T_t = \begin{bmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{bmatrix}, \qquad R_t = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad Q_t = \sigma^2$$

- Z_t and $H_t = 0$ imply that $\alpha_{1t} = y_t$;
- First state equation implies $y_{t+1} = \phi_1 y_t + \alpha_{2t} + \zeta_t$ with $\zeta_t \sim \mathcal{NID}(0, \sigma^2)$;
- Second state equation implies $\alpha_{2,t+1} = \phi_2 y_t$;

ARMA in State Space Form

Example: MA(1) model $y_{t+1} = \zeta_t + \theta\zeta_{t-1}$, in state space:

$$\alpha_{t+1} = T_t\alpha_t + R_t\zeta_t, \qquad \zeta_t \sim \mathcal{NID}(0, Q_t),$$
$$y_t = Z_t\alpha_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, H_t).$$

with 2×1 state vector α_t and system matrices:

$$Z_t = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad H_t = 0$$
$$T_t = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad R_t = \begin{bmatrix} 1 \\ \theta \end{bmatrix}, \qquad Q_t = \sigma^2$$

- Z_t and $H_t = 0$ imply that $\alpha_{1t} = y_t$;
- First state equation implies $y_{t+1} = \alpha_{2t} + \zeta_t$ with $\zeta_t \sim \mathcal{NID}(0, \sigma^2)$;
- Second state equation implies $\alpha_{2,t+1} = \theta\zeta_t$;

ARMA in State Space Form

Example: ARMA(2,1) model

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \zeta_t + \theta \zeta_{t-1}$$

in state space form

$$\alpha_t = \begin{bmatrix} y_t \\ \phi_2 y_{t-1} + \theta \zeta_t \end{bmatrix}$$

$$Z_t = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad H_t = 0,$$

$$T_t = \begin{bmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{bmatrix}, \qquad R_t = \begin{bmatrix} 1 \\ \theta \end{bmatrix}, \qquad Q_t = \sigma^2$$

All ARIMA(p, d, q) models have a (non-unique) state space representation.

UC models in State Space Form

State space model: $\alpha_{t+1} = T_t \alpha_t + R_t \zeta_t, \quad y_t = Z_t \alpha_t + \varepsilon_t$.

LL model $\Delta\mu_{t+1} = \eta_t$ and $y_t = \mu_t + \varepsilon_t$:

$$\begin{aligned} \alpha_t &= \mu_t, & T_t &= 1, & R_t &= 1, & Q_t &= \sigma_\eta^2, \\ & & Z_t &= 1, & & & H_t &= \sigma_\varepsilon^2. \end{aligned}$$

LLT model $\Delta\mu_{t+1} = \beta_t + \eta_t, \quad \Delta\beta_{t+1} = \xi_t$ and $y_t = \mu_t + \varepsilon_t$:

$$\alpha_t = \begin{bmatrix} \mu_t \\ \beta_t \end{bmatrix}, \quad T_t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad R_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_t = \begin{bmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\xi^2 \end{bmatrix},$$

$$Z_t = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad H_t = \sigma_\varepsilon^2.$$

UC models in State Space Form

State space model: $\alpha_{t+1} = T_t\alpha_t + R_t\zeta_t, \quad y_t = Z_t\alpha_t + \varepsilon_t$.

LLT model with season: $\Delta\mu_{t+1} = \beta_t + \eta_t, \quad \Delta\beta_{t+1} = \xi_t$, $S(L)\gamma_{t+1} = \omega_t$ and $y_t = \mu_t + \gamma_t + \varepsilon_t$:

$$\alpha_t = \begin{bmatrix} \mu_t & \beta_t & \gamma_t & \gamma_{t-1} & \gamma_{t-2} \end{bmatrix}',$$

$$T_t = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad Q_t = \begin{bmatrix} \sigma_\eta^2 & 0 & 0 \\ 0 & \sigma_\xi^2 & 0 \\ 0 & 0 & \sigma_\omega^2 \end{bmatrix}, \quad R_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Z_t = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad H_t = \sigma_\varepsilon^2.$$

Kalman Filter

- The Kalman filter calculates the mean and variance of the unobserved state, given the observations.
- The state is Gaussian: the complete distribution is characterized by the mean and variance.
- The filter is a recursive algorithm; the current best estimate is updated whenever a new observation is obtained.
- To start the recursion, we need a_1 and P_1, which we assumed given.
- There are various ways to initialize when a_1 and P_1 are unknown, which we will not discuss here.

Kalman Filter

The unobserved state α_t can be estimated from the observations with the *Kalman filter*:

$$\begin{aligned} v_t &= y_t - Z_t a_t, \\ F_t &= Z_t P_t Z_t' + H_t, \\ K_t &= T_t P_t Z_t' F_t^{-1}, \\ a_{t+1} &= T_t a_t + K_t v_t, \\ P_{t+1} &= T_t P_t T_t' + R_t Q_t R_t' - K_t F_t K_t', \end{aligned}$$

for $t = 1, \ldots, n$ and starting with given values for a_1 and P_1.

- Writing $Y_t = \{y_1, \ldots, y_t\}$,

$$a_{t+1} = \mathrm{E}(\alpha_{t+1}|Y_t), \qquad P_{t+1} = \mathrm{var}(\alpha_{t+1}|Y_t).$$

Kalman Filter

State space model: $\alpha_{t+1} = T_t \alpha_t + R_t \zeta_t, \quad y_t = Z_t \alpha_t + \varepsilon_t$.

- Writing $Y_t = \{y_1, \ldots, y_t\}$, define

$$a_{t+1} = \mathrm{E}(\alpha_{t+1}|Y_t), \qquad P_{t+1} = \mathrm{var}(\alpha_{t+1}|Y_t);$$

- The prediction error is

$$\begin{aligned} v_t &= y_t - \mathrm{E}(y_t|Y_{t-1}) \\ &= y_t - \mathrm{E}(Z_t \alpha_t + \varepsilon_t|Y_{t-1}) \\ &= y_t - Z_t \, \mathrm{E}(\alpha_t|Y_{t-1}) \\ &= y_t - Z_t a_t; \end{aligned}$$

- It follows that $v_t = Z_t(\alpha_t - a_t) + \varepsilon_t$ and $\mathrm{E}(v_t) = 0$;
- The prediction error variance is $F_t = \mathrm{var}(v_t) = Z_t P_t Z_t' + H_t$.

Lemma

The proof of the Kalman filter uses a lemma from multivariate Normal regression theory.

Lemma Suppose x, y and z are jointly Normally distributed vectors with $\mathrm{E}(z) = 0$ and $\Sigma_{yz} = 0$. Then

$$\mathrm{E}(x|y,z) = \mathrm{E}(x|y) + \Sigma_{xz}\Sigma_{zz}^{-1}z,$$

$$\mathrm{var}(x|y,z) = \mathrm{var}(x|y) - \Sigma_{xz}\Sigma_{zz}^{-1}\Sigma_{xz}',$$

Multivariate local level model

Seemingly Unrelated Time Series Equations model:

$$y_t = \mu_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, \Sigma_\varepsilon),$$
$$\mu_{t+1} = \mu_t + \eta_t, \qquad \eta_t \sim \mathcal{NID}(0, \Sigma_\eta).$$

- Observations are $p \times 1$ vectors;
- The disturbances ε_t, η_s are independent for all s, t;
- The p different time series are related through correlations in the disturbances.

For a full discussion, see Harvey and Koopman (1997).

A pdf version (scanned) at `http://staff.feweb.vu.nl/koopman` under section “Publications” and subsection “Published articles as contributions to books”.

Multivariate LL Model

The multivariate LL model is given by

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, & \varepsilon_t &\sim \mathcal{NID}(0, \Sigma_\varepsilon), \\ \mu_{t+1} &= \mu_t + \eta_t, & \eta_t &\sim \mathcal{NID}(0, \Sigma_\eta). \end{aligned}$$

- First difference
$$\Delta y_t = \eta_{t-1} + \Delta \varepsilon_t$$
is stationary;
- Reduced form: Δy_t is VMA(1) or VAR(∞);

Multivariate LL Model

- Stochastic properties are multivariate analogous of univariate case:

$$\begin{aligned}\Gamma_0 = \mathsf{E}(\Delta y_t\,\Delta y_t') &= \Sigma_\eta + 2\Sigma_\varepsilon\\ \Gamma_1 = \mathsf{E}(\Delta y_t\,\Delta y_{t-1}') &= -\Sigma_\varepsilon\\ \Gamma_\tau = \mathsf{E}(\Delta y_t\,\Delta y_{t-\tau}') &= 0, \qquad\qquad \tau \geq 2,\end{aligned}$$

- The unrestricted vector MA(1) process has $p^2 + p(p+1)/2$ parameters, the SUTSE has $p \times (p+1)$;
- Such multivariate reduced form representations can also be established for general models.

Homogeneous Multivariate LL Model

The homogeneous multivariate LL model is given by

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, \qquad & \varepsilon_t &\sim \mathcal{NID}(0, \Sigma_\varepsilon), \\ \mu_{t+1} &= \mu_t + \eta_t, \qquad & \eta_t &\sim \mathcal{NID}(0, q\Sigma_\varepsilon), \end{aligned}$$

where q is a non-negative scalar. This implies that $\Sigma_\eta = q\Sigma_\varepsilon$.

- The model is restricted, all series in y_t have the same dynamic properties (the same acf).
- Not so relevant in practical work apart from forecasting. It is the model representation for exponentially weighted moving average (EWMA) forecasting of multiple time series.
- This can be generalised to more general components models.
- Easy to estimate, only a set of univariate Kalman filters are required.

Common Levels

The common local level model is given by

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, \qquad & \varepsilon_t &\sim \mathcal{NID}(0, \Sigma_\varepsilon), \\ \mu_{t+1} &= \mu_t + \eta_t, \qquad & \eta_t &\sim \mathcal{NID}(0, \Sigma_\eta), \end{aligned}$$

where rank$(\Sigma_\eta) = r < p$.

- the model can be described by r underlying level components, the *common levels*;
- $$\Sigma_\eta = A \Sigma_c A',$$
  A is $p \times r$, Σ_c is $r \times r$ of full rank;
- interpretation of A: factor loading matrix.

Common Levels

The common local level model

$$
\begin{aligned}
y_t &= \mu_t + \varepsilon_t, \qquad & \varepsilon_t &\sim \mathcal{NID}(0, \Sigma_\varepsilon), \\
\mu_{t+1} &= \mu_t + \eta_t, \qquad & \eta_t &\sim \mathcal{NID}(0, A\Sigma_c A'),
\end{aligned}
$$

can be rewritten in terms of underlying levels:

$$
\begin{aligned}
y_t &= a + A\mu_t^c + \varepsilon_t, \\
\mu_{t+1}^c &= \mu_t^c + \eta_t^c, \qquad \eta_t^c \sim \mathcal{NID}(0, \Sigma_c),
\end{aligned}
$$

so that

$$
\mu_t = a + A\mu_t^c, \qquad \eta_t = A\eta_t^c.
$$

Common Levels

For the common local level model

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, & \varepsilon_t &\sim \mathcal{NID}(0, \Sigma_\varepsilon), \\ \mu_{t+1} &= \mu_t + \eta_t, & \eta_t &\sim \mathcal{NID}(0, A\Sigma_c A'), \end{aligned}$$

notice that

- decomposition $\Sigma_\eta = A\Sigma_c A'$ is not unique;
- identification restrictions: Σ_c is diagonal, Choleski decomposition, principal compoments (based on economic theory);
- more interesting interpretation can be obtained by factor rotations;
- can be interpreted as *dynamic* factor analysis, see later.

Common components

Common dynamic factors:

- are useful for interpretation $\rightarrow$ cointegration;
- have consequence for inference and forecasting (dimension of parameter space reduces as a result).
- common local level model can be generally represented as a VAR(∞) or VECM models, details can be provided upon request.

Multivariate components

- So far, we have concentrated on multivariate variants of the local level model;
- Similar considerations can be applied to other components such as the slope of the trend, seasonal and cycle components and other time-varying features in the multiple time series.
- Harvey and Koopman (1997) review such extensions.
- In particular, they define the similar cycle component, see Exercises.

Common and idiosyncratic factors

Multiple trends can also be decomposed into a one common factor and multiple idiosyncratic factors:

$$y_t = \mu_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, \Sigma_\varepsilon),$$
$$\mu_{t+1} = \mu_t + \eta_t, \qquad \eta_t \sim \mathcal{NID}(0, \Sigma_\eta),$$

where $\Sigma_\eta = \delta\delta' + D_\eta$ with vector δ and diagonal matrix D_η. This implies that the level can be represented by

$$\mu_t = \delta\mu_t^c + \mu_t^*, \qquad \eta_t = \delta\eta_t^c + \eta_t^*$$

with common level (scalar) μ_t^c and ”independent” level μ_{it}^* generated by

$$\Delta\mu_{t+1}^c = \eta_t^c \sim \mathcal{NID}(0, 1), \qquad \Delta\mu_{t+1}^* = \eta_t^* \sim \mathcal{NID}(0, D_\eta).$$

Mulitvariate Kalman filter

The Kalman filter is valid for the general multivariate state space model.

Computationally it is not convenient when p becomes large, very large.

Each step of the Kalman filter requires the inversion of the $p \times p$ matrix F_t. This is no problem when $p = 1$ (univariate) but when $p > 20$, say, it will slow down the Kalman filter considerably.

However, we can treat each element in the $p \times 1$ observation vector y_t as a single realisation. In other words, we can "update" each single element of y_t within the Kalman filter.

The arguments are given in DK book §6.4.

The same applies to smoothing.

Univariate treatment of Kalman filter

- Consider standard model: $y_t = Z_t \alpha_t + \varepsilon_t$ and $\alpha_{t+1} = T_t \alpha_t + R_t \eta_t$ where $\text{Var}(\varepsilon_t) = H_t$ is diagonal.
- Observation vector $y_t = (y_{t,1}, \ldots, y_{t,p_t})'$ is treated and we view observation model as a set of p_t separate equations.
- We then have, $y_{t,i} = Z_{t,i}\alpha_{t,i} + \varepsilon_{t,i}$ with $\alpha_{t,i} = \alpha_t$ for $i = 1, \ldots, p_t$.
- The associated transition equations become $\alpha_{t,i+1} = \alpha_{t,i}$ for $i = 1, \ldots, p_t$ and $\alpha_{t+1,1} = T_t \alpha_{t,p_t} + R_t \eta_t$ for $t = 1, \ldots, n$.
- This disentangled model can be treated by the Kalman filter and smoother equations straightforwardly.
- Innovations are now relative to the past and the "previous" observations inside y_{t,p_t} !
- Non-diagonal matrix H_t can be treated by data-transformation or by including ε_t in the state vector α_t.
- More details in DK book §6.4.

Exercise 1

Consider the common trends model of Harvey and Koopman (1997, §§9.4.1 and 9.4.2).

1. Put the common trends model with (possibly common) stochastic slopes and based on equations (21)-(23) in state space form.
2. Put the common trends model with (possibly common) stochastic slopes and based on equations (24)-(26) in state space form. Define all vectors and matrices precisely.
3. Discuss the generalisation of $\Sigma_\eta \neq 0$ and the consequences for the state space formulation of the model as in 2.

Exercise 2

Consider the multivariate trend model of Harvey and Koopman (1997).

1. Consider a multiple data set of N time series y_t. The aim is to decompose the time series into trend and stationary components. It is further required that the multiple trend can be decomposed into a common single trend (common to all N time series) and idiosyncratic trends (specific to the individual time series).
    - Formulate a model for such a decomposition.
    - Discuss the identification of the different trends.
    - Express the model in state space form.
2. Once multiple trend models are expressed in state space form, we need to estimate the parameter coefficients of the model. Please describe shortly some relevant issues of maximum likelihood estimation. Is it feasible ? What problems can you expect ? Any recommendations for a successful implementation ?

Exercise 3

Consider the similar cycle model of Harvey and Koopman (1997) with observation equation

$$y_t = \psi_t + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \Sigma_\varepsilon),$$

where y_t is a 3×1 observation vector. Cycle ψ_t represents a common similar cycle component of rank 2.

1. Please provide the state space representation of this model.
2. Comment on the restrictive nature of the similar cycle model.
3. How would you modify the similar cycle model so that each time series in y_t has a different cycle frequency λ.
4. Can you apply the univariate Kalman filter of DK §6.4 in case Σ_ε is diagonal ? What if Σ_ε is not diagonal ? Give details.