Observation Driven Mixed-Measurement Dynamic Factor Models with an Application to Credit Risk

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Motivation

- Economic time series often share common features, e.g. business cycle dynamics.

- Economic time series may be continuous and/or discrete and be observed at different frequencies.

- Credit risk application:
  1. Credit conditions depend on the macroeconomy
  2. Corporate defaults tend to “cluster” together in time.
  3. Default probabilities are higher than can be explained by covariates.
Contributions

- We introduce observation-driven mixed measurement panel data models.
- The approach allows for non-linear, non-Gaussian models with common factor across different distributions.
- Credit risk application: we develop new models for credit ratings transitions and loss-given-default (LGDs).
- The models include:
  1. Time-varying ordered logit
  2. Time-varying beta distribution
Mixed measurement panel data models

\[ y_{it} \sim p_i(y_{it} | F^t, Y^{t-1}; \psi), \quad i = 1, \ldots, N, \]
\[ f_{t+1} = \omega + B_1 f_t + A_1 s_t \]

with log-likelihood function

\[ \log p(y_t | F^t, Y^{t-1}; \psi) = \sum_{i=1}^{N} \delta_{it} \log p_i(y_{it} | F^t, Y^{t-1}; \psi) \]

- The observation vector \( y_t \) is \( N \times 1 \).
- The individual observations \( y_{it} \) may come from different distributions.
- **KEY**: The factors \( f_t \) may be common across distributions.
- \( \delta_{it} \) is an indicator function equal to 1 if \( y_{it} \) is observed and zero otherwise. Missing values are naturally taken into account.
Mixed measurement panel data models

\[ y_{it} \sim p_i(y_{it} | F^t, Y^{t-1}; \psi), \quad i = 1, \ldots, N, \]
\[ f_{t+1} = \omega + B_1 f_t + A_1 s_t \]

The score function is

\[ s_t = S_t \nabla_t \]
\[ \nabla_t = \sum_{i=1}^{N} \delta_{it} \nabla_{i,t} = \sum_{i=1}^{N} \delta_{it} \frac{\partial \log p_i(y_{it} | F^t, Y^{t-1}; \psi)}{\partial f_t}, \]

- **KEY**: The score function allows us to pool information from different observations to estimate the common factor \( f_t \).
- The score function also allows us to “weight” the information coming from different types of data.
Consider the eigenvalue-eigenvector decomposition of Fisher’s (conditional) information matrix

\[ \mathcal{I}_t = E_{t-1}[\nabla_t \nabla_t'] = U_t \Sigma_t U_t', \]

We define the scaling matrix as

\[ S_t = U_t \Sigma_t^{-1/2} U_t'. \]

- \( S_t \) is then the “square root” of a generalized inverse.
- The innovations \( s_t \) driving \( f_t \) have an identity covariance matrix, when the info. matrix is non-singular.
- The conditional information matrix is additive for our models:

\[ \mathcal{I}_t = E_{t-1}[\nabla_t \nabla_t'] = \sum_{i=1}^{N} \delta_{it} E_{i,t-1}[\nabla_{it} \nabla_{it}']. \]
The log-likelihood function for an observation-driven model can easily be computed.

The ML estimator is

\[ \hat{\psi} = \arg \max_\psi \sum_{t=1}^T \sum_{i=1}^N \delta_{it} \log p_i(y_{it}|F^t, Y^{t-1}; \psi), \]

Estimation is similar to a GARCH model.

For a given value of \( \psi \), the factors are computed \( \{f_0, f_1, \ldots, f_T\} \) from the recursion:

\[ f_{t+1} = \omega + B_1 f_t + A_1 s_t \]
Growing econometrics literature on models for credit risk: McNeil et al. (2005), Bauwens and Hautsch (JFEct, 2006), Gagliardini and Gourieroux (JFEct, 2005), Koopman Lucas and Monteiro (JEct, 2008), Duffie et al. (JFE, JoF 2008).

Basic observations:
1. Probability of default varies over time with the business cycle.
2. Conditional on default, the loss (recovery rate) varies with the business cycle.
3. We observe excess clustering of defaults and ratings transitions beyond what can be explained by simply adding covariates.
4. The literature focuses on a credit risk or frailty factor.

Industry standard models are too simple to capture these features.

New models in the literature are parameter driven models requiring simulation methods for estimation.

We provide observation driven alternatives.
We observe data from Jan. 1980 to March 2010.

7,505 companies are rated by Moody’s.

We pool these into 5 ratings categories (IG, BB, B, C, D).

We observe transitions, e.g. IG → BB or C → D

There are $J = 16$ total types of transitions.

19,450 total credit rating transitions.

1,342 transitions are defaults.

1,125 measurements of loss-given default (LGD).

LGD is the fraction of principal an investor loses when a firm defaults.

We also observe six macroeconomic variables: industrial production growth, credit spread, unemployment, annual S&P500 returns, realized volatility, real GDP growth (qtly).
Models

- Credit ratings can be modeled using the (static) ordered probit model of CreditMetrics; one of the current industry standards, see Gupton Stein (2005).
- LGD’s are often modeled by (static) beta distributions.
- GOAL: Build models that improve on current industry standards and are (relatively) easy to implement and estimate.
  1. Time-varying ordered logit
  2. Time-varying beta distribution
- Forecasting credit risk.
- Simulation of loss distributions and scenario analysis.
- Bank executives and regulators and can use them for “stress testing.”
Mixed measurement model for credit risk

\[ y_t^m \sim N(\mu_t, \Sigma_m) \]
\[ y_{i,t}^c \sim \text{Ordered Logit} (\pi_{ijt}, j \in \{IG, BB, B, C, D\}) \]
\[ y_{k,t}^r \sim \text{Beta} (a_{kt}, b_{kt}), \quad k = 1, \ldots, K_t, \]

- \( y_t^m \) are the macro variables.
- \( y_{i,t}^c \) are indicator variables for each credit rating \( j \) for firm \( i \).
- \( y_{k,t}^r \) are the LGDs for the \( k \)-th default.
- \( K_t \) are the number of defaults in period \( t \).
- \( \mu_t, \pi_{ijt}, \) and \( (a_{kt}, b_{kt}) \) are functions of an \( M \times 1 \) vector of factors \( f_t \).
Time varying Gaussian model for macro data

\[ y_t^m \sim N(\mu_t, \Sigma_m), \]
\[ \mu_t = Z^m f_t. \]

- \( Z^m \) is a \((6 \times M)\) matrix of factor loadings.
- \( \Sigma_m \) is a \((6 \times 6)\) diagonal covariance matrix.
- \( \tilde{S}_t \) is a selection matrix indicating which macro variables are observed at time \( t \).

\[ \nabla_t^m = (\tilde{S}_t Z^m)' \left( \tilde{S}_t \Sigma_m \tilde{S}_t' \right)^{-1} \tilde{S}_t (y_t^m - \mu_t), \]
\[ \mathcal{I}_t^m = (\tilde{S}_t Z^m)' \left( \tilde{S}_t \Sigma_m \tilde{S}_t' \right)^{-1} \tilde{S}_t Z^m. \]
Moody’s monthly credit ratings transitions

The data have been pooled together each month.
Time-varying ordered logit

\[ y_{i,t}^c \sim \text{Ordered Logit} \left( \pi_{ijt}, j \in \{IG, BB, B, C, D\} \right), \]

\[ \pi_{ijt} = P \left[ R_{i,t+1} = j \right] = \tilde{\pi}_{ijt} - \tilde{\pi}_{i,j-1,t}, \]

\[ \tilde{\pi}_{ijt} = P \left[ R_{i,t+1} \leq j \right] = \frac{\exp(\theta_{ijt})}{1 + \exp(\theta_{ijt})}, \]

\[ \theta_{ijt} = z_{ijt}^c - Z_{it}^c f_t. \]

- \( y_{it}^c \) is an indicator variable for each rating type.
- \( J^c = 5 \) categories \( j \in \{IG, BB, B, C, D\} \).
- \( R_{it} \) is the rating for firm \( i \) at the start of month \( t \).
- \( \pi_{ijt} \) is the probability that firm \( i \) is in category \( j \).
- \( \tilde{\pi}_{i,D,t} = 0 \) and \( \tilde{\pi}_{i,IG,t} = 1 \).
- To our knowledge, a time-varying ordered logit model is new.
Time-varying ordered logit

The contribution to the log-likelihood at time $t$ is

$$\ln p_i(y_{it}^c|F^t, Y^{t-1}; \psi) = \sum_{i=1}^{N_t} \sum_{j=1}^{J^c} y_{ijt}^c \log (\pi_{ijt})$$

The score and information matrices are

$$\nabla_t^c = - \sum_{i=1}^{N_t} \sum_{j=1}^{J^c} \frac{y_{ijt}^c}{\pi_{ijt}} \cdot \dot{\pi}_{ijt} \cdot Z_{it}^c,$$

$$\mathcal{I}_t^c = \sum_{i=1}^{N_t} n_{it} \left( \sum_j \frac{\dot{\pi}_{ij,t}^2}{\pi_{ij,t}} \right) Z_{it}^c Z_{it}^{c'}$$

where

$$\dot{\pi}_{ijt} = \ddot{\pi}_{ijt} (1 - \ddot{\pi}_{ijt}) - \ddot{\pi}_{i,j-1,t} (1 - \ddot{\pi}_{i,j-1,t}).$$
When a firm defaults, investors typically lose a fraction of their investment (alternatively, they recover a fraction of their investment).

The fraction of losses experienced by investors also varies with the business cycle.

We develop a new model for a time-varying beta distribution.

See McNeil and Wendin (2007 JEmpFin) for Bayesian inference in a state space model.
Loss given default by transition type

![Graph showing loss given default by transition type](image.png)
Time-varying beta distribution

\[
y_{k,t} \sim \text{Beta} \left( a_{kt}, b_{kt} \right), \quad k = 1, \ldots, K_t,
\]

\[
a_{kt} = \beta_r \cdot \mu_{kt}
\]

\[
b_{kt} = \beta_r \cdot (1 - \mu_{kt})
\]

\[
\log \left( \frac{\mu_{kt}}{1 - \mu_{kt}} \right) = z^r + Z^r f_t.
\]

- We observe \( K_t \geq 0 \) defaults at time \( t \).
- \( 0 < y_{k,t}^r < 1 \) is the amount lost conditional on the \( k \)-th default.
- \( \mu_{kt}^r \) is the mean of the beta distribution.
- \( z^r \) is the unconditional level of LGDs.
- \( Z^r \) is a \( (1 \times M) \) vector of factor loadings.
- \( \beta_r \) is a scalar parameter.
**Time-varying beta distribution**

The contribution to the log-likelihood at time $t$ is

$$
\ln p_i(y_{kt}^r | F^t, Y^{t-1}; \psi) = \sum_{k=1}^{K_t} (a_{kt} - 1) \log (y_{kt}^r) + (b_{kt} - 1) \log (1 - y_{kt}^r)
- \log [B (a_{kt}, b_{kt})]
$$

The score and information matrices are

$$
\nabla_t^r = \beta_r \sum_{k=1}^{K_t} \mu_{kt}^r (1 - \mu_{kt}^r) (Z^r)' (1, -1) \left( \log(y_{kt}^r), \log(1 - y_{kt}^r) \right)' - \dot{B} (a_{kt}, b_{kt})
$$

$$
\mathcal{I}_t^r = \beta_r \sum_{k=1}^{K_t} (\mu_{kt}^r (1 - \mu_{kt}^r))^2 (Z^r)' (1, -1) \left( \dot{B} (a_{kt}, b_{kt}) \right) (1, -1)' Z^r
$$

where

$$
\sigma_{kt}^2 = \mu_{kt}^r \cdot (1 - \mu_{kt}^r)/(1 + \beta_r).
$$
The macro data $y_t^m$ has been standardized.

We consider models with $p = 1$ and $q = 1$ factor dynamics.

For identification of the level parameters, we set $ω = 0$ in the factor recursion:

$$f_{t+1} = A_1 s_t + B_1 f_t$$

For identification of the factors, we also impose restrictions on $Z^m$, $Z^c$, and $Z^r$.

Some parameters have been pooled for “rare” transitions; e.g., IG $\rightarrow$ D and BB $\rightarrow$ D.

Moody’s re-defined several categories in April 1982 and Oct. 1999 causing incidental re-ratings (outliers), which we handle via dummy variables for these dates.
AIC, BIC, and log-likelihoods for different models

<table>
<thead>
<tr>
<th>(3,0,0)</th>
<th>(3,1,0)</th>
<th>(3,2,0)</th>
<th>(3,2,1)</th>
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<td>-40056.2</td>
<td>-39817.1</td>
<td>-39780.8</td>
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<td>AIC</td>
<td>80242.4</td>
<td>79776.2</td>
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<td>BIC</td>
<td>80991.0</td>
<td>80594.0</td>
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<th>(4,0,0)</th>
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<td>log-Lik</td>
<td>-39828.7</td>
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</tr>
<tr>
<td>AIC</td>
<td>79805.3</td>
<td>79352.7</td>
<td><strong>79293.2</strong></td>
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<td>BIC</td>
<td>80658.0</td>
<td>80274.0</td>
<td><strong>80273.0</strong></td>
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The number of factors for each data type are represented by \((m, c, r)\).
Parameter estimates for the (4,2,0) model

Factor dynamics and macro loadings $Z^m$

<table>
<thead>
<tr>
<th></th>
<th>macro₁</th>
<th>macro₂</th>
<th>macro₃</th>
<th>macro₄</th>
<th>frailty₁</th>
<th>frailty₂</th>
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<tr>
<td>A</td>
<td>0.221***</td>
<td>0.154***</td>
<td>0.300***</td>
<td>0.282***</td>
<td>0.033***</td>
<td>0.036***</td>
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<tr>
<td></td>
<td>(0.011)</td>
<td>(0.014)</td>
<td>(0.016)</td>
<td>(0.023)</td>
<td>(0.006)</td>
<td>(0.010)</td>
</tr>
<tr>
<td>B</td>
<td>0.966***</td>
<td>0.974***</td>
<td>0.924***</td>
<td>0.896***</td>
<td>0.974***</td>
<td>0.981***</td>
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<tr>
<td></td>
<td>(0.012)</td>
<td>(0.016)</td>
<td>(0.022)</td>
<td>(0.026)</td>
<td>(0.012)</td>
<td>(0.012)</td>
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</table>

$Z^m$

<table>
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<tr>
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<th>macro₁</th>
<th>macro₂</th>
<th>macro₃</th>
<th>macro₄</th>
<th>frailty₁</th>
<th>frailty₂</th>
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<tr>
<td>IP</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>UR</td>
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<td>(0.061)</td>
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<td>RGDP</td>
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<td>0.000</td>
<td>0.295***</td>
<td>0.000</td>
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<td></td>
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<td>(0.120)</td>
<td></td>
<td>(0.068)</td>
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<tr>
<td>Cr.Spr.</td>
<td>-0.275***</td>
<td>0.000</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>(0.075)</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>$r_{S&amp;P}$</td>
<td>0.000</td>
<td>-0.358**</td>
<td>-0.293***</td>
<td>1.179***</td>
<td>0.000</td>
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<tr>
<td></td>
<td></td>
<td>(0.157)</td>
<td>(0.076)</td>
<td>(0.086)</td>
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<tr>
<td>$\sigma_{S&amp;P}$</td>
<td>0.101*</td>
<td>0.245*</td>
<td>0.563***</td>
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<td>(0.096)</td>
<td>(0.176)</td>
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Parameter estimates for the $(4,2,0)$ model

Credit rating and LGD loadings $Z^c$ and $Z^r$

<table>
<thead>
<tr>
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<th>macro$^1$</th>
<th>macro$^2$</th>
<th>macro$^3$</th>
<th>macro$^4$</th>
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<th>frailty$^2$</th>
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<tr>
<td>IG</td>
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<td>0.000</td>
<td>0.217***</td>
<td>-0.110*</td>
<td>1.520***</td>
<td>-0.727**</td>
</tr>
<tr>
<td></td>
<td>(0.051)</td>
<td></td>
<td>(0.053)</td>
<td>(0.064)</td>
<td>(0.283)</td>
<td>(0.270)</td>
</tr>
<tr>
<td>BB</td>
<td>0.000</td>
<td>0.204***</td>
<td>0.158***</td>
<td>-0.077*</td>
<td>1.000</td>
<td>0.000</td>
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<tr>
<td></td>
<td></td>
<td>(0.057)</td>
<td>(0.038)</td>
<td>(0.041)</td>
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</tr>
<tr>
<td>B</td>
<td>-0.154***</td>
<td>0.130**</td>
<td>0.150***</td>
<td>-0.121***</td>
<td>0.914***</td>
<td>0.000</td>
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<tr>
<td></td>
<td>(0.036)</td>
<td>(0.056)</td>
<td>(0.031)</td>
<td>(0.038)</td>
<td>(0.137)</td>
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<tr>
<td>CCC</td>
<td>-0.283***</td>
<td>0.000</td>
<td>0.076*</td>
<td>0.000</td>
<td>1.486***</td>
<td>1.000</td>
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<tr>
<td></td>
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<td></td>
<td>(0.048)</td>
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</table>

<table>
<thead>
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<th>macro$^4$</th>
<th>frailty$^1$</th>
<th>frailty$^2$</th>
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<tbody>
<tr>
<td>$Z^r$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.274***</td>
<td>-0.077*</td>
<td>0.938**</td>
<td>0.913***</td>
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<tr>
<td></td>
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<td></td>
<td>(0.045)</td>
<td>(0.057)</td>
<td>(0.315)</td>
<td>(0.199)</td>
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</table>
Estimated factors for the (4,2,0) model
Time-varying transition probabilities

- IG to IG
  - 1980: 0.9950
  - 1990: 0.9975
  - 2000: 1.0000
- IG to BB
  - 1980: 0.002
  - 1990: 0.004
  - 2000: 0.006
- IG to B
  - 1980: 0.00025
  - 1990: 0.00050
  - 2000: 0.00075
- IG to CCC
  - 1980: 5.00e-5
  - 1990: 0.0001
  - 2000: 0.00015
- IG to Default
  - 1980: 5.00e-5
  - 1990: 0.0001
  - 2000: 0.00015
- BB to IG
  - 1980: 0.0025
  - 1990: 0.0050
  - 2000: 0.0075
- BB to BB
  - 1980: 0.975
  - 1990: 0.980
  - 2000: 0.985
- BB to B
  - 1980: 0.010
  - 1990: 0.015
  - 2000: 0.020
- BB to CCC
  - 1980: 0.00025
  - 1990: 0.00050
  - 2000: 0.00075
- BB to Default
  - 1980: 5.00e-5
  - 1990: 0.0001
  - 2000: 0.00015
- B to IG
  - 1980: 0.0005
  - 1990: 0.0010
- B to BB
  - 1980: 0.0025
  - 1990: 0.0050
  - 2000: 0.0075
- B to B
  - 1980: 0.97
  - 1990: 0.98
  - 2000: 0.99
- B to CCC
  - 1980: 0.01
  - 1990: 0.02
  - 2000: 0.03
- B to Default
  - 1980: 0.00025
  - 1990: 0.00050
  - 2000: 0.00075
  - 2010: 0.0100
- CCC to IG
  - 1980: 0.0001
  - 1990: 0.0002
  - 2000: 0.0003
- CCC to BB
  - 1980: 0.0002
  - 1990: 0.0004
  - 2000: 0.0006
- CCC to B
  - 1980: 0.005
  - 1990: 0.010
  - 2000: 0.015
- CCC to CCC
  - 1980: 0.925
  - 1990: 0.950
  - 2000: 0.975
- CCC to Default
  - 1980: 0.025
  - 1990: 0.050
  - 2000: 0.075
  - 2010: 0.100
Top and bottom left are loss distributions. Top right is a plot of the mean through time. Bottom right are transition probabilities from BB → D.
Simulating cumulative loss distributions

- Most financial institutions carry a large portfolio of credit related securities.
- Given a portfolio at time $T$, we can use the models to simulate different possible risk scenarios.
- GOAL: determine the amount of capital banks may need in the future.
- What happens if we do not include time-varying parameters $f_t$ in the model?
- Scenario analysis:
  1. What happens if there is a negative shock to RGDP?
  2. What happens if there is an increase to credit spreads?
Simulating cumulative loss distributions

- At time $T$, a financial institution holds a portfolio of bonds.
- The goal is to forecast the loss distribution at time $T + h$.
- We assume a portfolio of firms with 1144 firms rated IG, 265 firms rated BB, 615 firms rated B, and 311 firms rated CCC.
- In the paper, we consider losses due only to default.
- For simplicity, we do not assume a time-varying discount function.
- We use 500,000 simulations.
- We can start at different values $f_T$. 
Simulating cumulative loss distributions

Cumulative losses on a portfolio of bonds at different horizons.

Comparison between different values of $f_T$ starting in a recession and expansion for the $(4,2,0)$ model.
Simulating cumulative loss distributions

Comparison of cumulative loss distributions with/without factors.

Left: starting at $f_T = 0$. Right: $f_T$ starting in a recession.
Conclusion and future work

- We introduce a new class of observation-driven models for mixed-measurement data which share exposure to common factors.
- Missing values and mixed frequencies are handled in a natural way.
- Using this approach, we develop new models for credit risk.
- The models can be used for simulating loss distributions, stress testing, and scenario analysis.

Future work:
- When computing loss distributions, current models do not account for changes in market prices of bonds or loans.
- Current models depend on industry credit ratings by Moody's, Fitch, Standard & Poors.
- Potential to use alternative sources of data.